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Kalkulus Peubah Banyak (B) Tugas Bab 4 (27 Oktober 2023)

5. $\int_{x=0}^{1}\int_{y=x}^{y=\exp(x)} y\, dy\, dx$

$= \int_{x=0}^{1}\left[\int_{x}^{\exp(x)} y\, dy\right] dx = \int_{0}^{1}\left[\frac{1}{2}y^2\right]_x^{e^x} dx$

$= \frac{1}{2}\int_0^1 (e^x)^2 - x^2\, dx$

$= \frac{1}{2}\int_0^1 e^{2x} - x^2\, dx$

$= \frac{1}{2}\left[\int_0^1 e^{2x}dx - \int_0^1 x^2 dx\right]$

$= \frac{1}{2}\left[\left[\frac{1}{2}e^{2x}\right]_0^1 - \left[\frac{1}{3}x^3\right]_0^1\right]$

$= \frac{1}{2}\left[\frac{1}{2}(e^2-1) - \frac{1}{3}(1-0)\right]$

$= \frac{1}{2}\left[\frac{1}{2}e^2 - \frac{1}{2} - \frac{1}{3}\right]$

$= \frac{1}{2}\left[\frac{1}{2}e^2 - \frac{5}{6}\right]$

$= \frac{1}{4}e^2 - \frac{5}{12} + C$ //

6. $\int_{y=1}^{2}\int_{x=\sqrt{y}}^{x=y^2} x\, dy\, dx = \int_1^2\int_{\sqrt{y}}^{y^2} x\, dx\, dy$

$= \int_1^2\left[\int_{\sqrt{y}}^{y^2} x\, dx\right] dy$

$= \int_1^2\left[\left[\frac{1}{2}x^2\right]_{\sqrt{y}}^{y^2}\right] dy$

$= \int_1^2 \frac{1}{2}\left[y^4 - y\right] dy$

$= \frac{1}{2}\left[\int_1^2 y^4 dy - \int_1^2 y\, dy\right]$

$= \frac{1}{2}\left[\left[\frac{1}{5}y^5\right]_1^2 - \left[\frac{1}{2}y^2\right]_1^2\right]$

$= \frac{1}{2}\left[\frac{32}{5} - \frac{1}{5} - 2 - \frac{1}{2}\right]$

$= \frac{1}{2}\left[\frac{31}{5} - \frac{5}{2}\right]$

$= \frac{1}{2}\left[\frac{37}{10}\right]$

$= \frac{37}{20} = 1\frac{17}{20}$ //

SiDU

8. C. $\iint_A xy\,dx\,dy$ jika A luasan di antara parabola $y = x^2$ dan garis $2x - y + 8 = 0$

$y = 2x + 8$

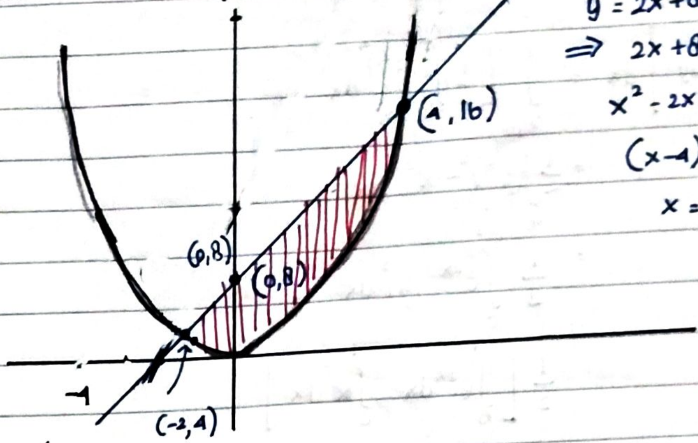
$\Rightarrow 2x + 8 = x^2$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4 \vee x = -2$

▶ Sketsa Daerah :



$y = x^2$

* $2x - y + 8 = 0$

- $x = 0 \rightarrow y = 8$
- $y = 0 \rightarrow x = -4$

$$\text{Luas} = \int_{-2}^{4}\int_{x^2}^{2x+8} xy\,dy\,dx = \int_{-2}^{4}\left[\int_{x^2}^{2x+8} xy\,dy\right]dx$$

$$= \int_{-2}^{4}\left[\left[\frac{x}{2}y^2\right]_{x^2}^{2x+8}\right]dx$$

$$= \int_{-2}^{4}\left[\frac{x}{2}\left((2x+8)^2 - x^4\right)\right]dx$$

$$= \frac{1}{2}\int_{-2}^{4}\left[x\left(4x^2+32x+64-x^4\right)\right]dx$$

$$= \frac{1}{2}\int_{-2}^{4}\left[4x^3+32x^2+64x-x^5\right]dx$$

$$= \frac{1}{2}\left[x^4+\frac{32}{3}x^3+32x^2-\frac{1}{6}x^6\right]_{-2}^{4}$$

$$= \frac{1}{2}\left[\left(4^4+\frac{32}{3}4^3+32.4^2-\frac{1}{6}4^6\right)-\left((-2)^4+\frac{32}{3}(-2)^3+32(-2)^2-\frac{1}{6}(-2)^6\right)\right]$$

$$= \frac{1}{2}\left[\left(256+\frac{32}{3}.64+32.16-\frac{1}{6}.4096\right)-\left(16-\frac{256}{3}+128-\frac{64}{6}\right)\right]$$

$$= \frac{1}{2}\left[\left(256+512-16-128\right)+\left(\frac{2048}{3}+\frac{256}{3}\right)-\left(\frac{4096}{6}+\frac{64}{6}\right)\right]$$

$$= \frac{1}{2}\left[624+\frac{2304}{3}-\frac{4160}{6}\right]$$

$$= \frac{1}{2}\left[624+\frac{4608-4032}{6}\right]$$

$$= \frac{1}{2}\left[624+\frac{576}{6}\right]$$

$$= \frac{1}{2}\left[624+96\right]$$

$$= \frac{720}{2}$$

$$= 360$$

SIDU

8. c). Jika A dibatasi parabola $y = x^2$ dan $2x - y + 8 = 0$ serta gambar 26, maka

▶ $2x - y + 8 = 0$

- $x = 0 \rightarrow y = 8$ $(0,8)$
- $y = 0 \rightarrow x = -4$ $(-4,0)$
- $y = 2 \rightarrow x = -3$ $(-3,2)$
- $y = 4 \rightarrow x = -2$ $(-2,4)$
- $x = \frac{\pi}{2} \rightarrow y = \pi + 8$ $(\frac{\pi}{2}, \pi+8) \Rightarrow (\frac{\pi}{2}, 11{,}14)$
- $x = \pi \rightarrow y = 2\pi + 8$ $(\pi, 2\pi+8)$ $(\pi, 14{,}28)$

$y = x^2 \rightarrow 2x - x^2 + 8 = 0$

$x^2 - 2x - 8 = 0$

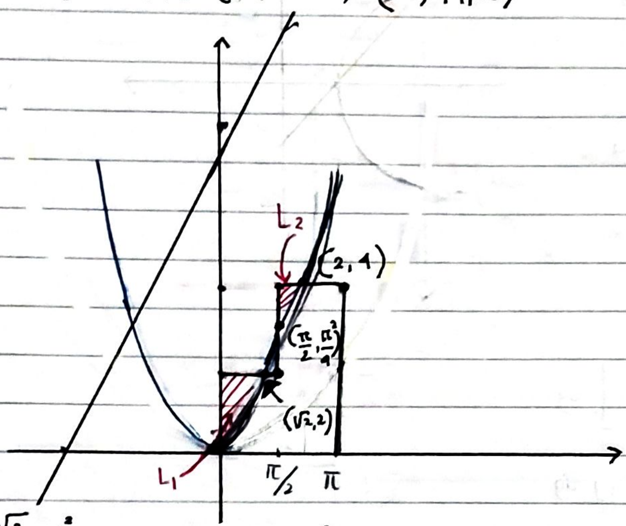
$(x-4)(x+2) = 0$

$x = 2 \vee x = -2$

▶ $y = x^2$

- $y = 2 \rightarrow x = \sqrt{2}$
- $y = 4 \rightarrow x = 2$

- $x = \frac{\pi}{2} \rightarrow y = \frac{\pi^2}{4}$ $(\frac{\pi}{2}, 2{,}47)$
- $x = \pi \rightarrow y = \pi^2$ $\rightarrow (\pi, 9{,}87)$



▶ $L_1 : \int_0^{\sqrt{2}} \int_2^{x^2} xy \, dy \, dx = \int_0^{\sqrt{2}} \left[\int_2^{x^2} xy \, dy \right] dx$

$= \int_0^{\sqrt{2}} \left[\frac{x}{2} y^2 \right]_2^{x^2} dx$

$= \int_0^{\sqrt{2}} \left[\frac{x}{2} x^4 - \frac{x}{2} \cdot 4 \right] dx$

$= \int_0^{\sqrt{2}} \left[\frac{x^5}{2} - 2x \right] dx$

$= \left[\frac{1}{12} x^6 \Big]_0^{\sqrt{2}} - x^2 \Big]_0^{\sqrt{2}} \right]$

$= \frac{1}{12} \cdot 8 - 2$

$= \frac{2}{3} - \frac{6}{3}$

$= -\frac{4}{3}$ //

▶ $L_2 = \int_{\pi/2}^{2} \int_{x^2}^{4} xy \, dy \, dx = \int_{\pi/2}^{2} \left[\int_{x^2}^{4} xy \, dy \right] dx$

$= \int_{\pi/2}^{2} \left[\frac{xy^2}{2} \Big]_{x^2}^{4} \right] dx$

$= \int_{\pi/2}^{2} \left[8x - \frac{x^5}{2} \right]$

$= 4x^2 \Big]_{\pi/2}^{2} - \frac{1}{8} x^4 \Big]_{\pi/2}^{2}$

$= 16 - 4\left[\frac{\pi}{2}\right]^2 - \left(\frac{1}{8} \cdot 16 - \frac{1}{8} \frac{\pi^4}{2^4} \right)$

$= 16 - 2 - \frac{4\pi^2}{4} + \frac{\pi^4}{128}$

$= 14 - \pi^2 + \frac{\pi^4}{128}$

$L_{tot} = -\frac{4}{3} + 14 - \pi^2 + \frac{\pi^4}{128}$

$= \frac{38}{3} - \pi^2 + \frac{\pi^4}{128}$

//

9. b. $\displaystyle\int_0^a\int_{a-y}^{\sqrt{a^2-y^2}} dx\,dy$, tentukan : a). Gambar daerah integrasinya

b). Hitung integralnya

c). Tukar urutan integrasinya dan hitung nilai integralnya

Jawab :

a). Gambar daerah integrasinya

- $x = \sqrt{a^2-y^2}$ • $y = a$
- $x = a - y$ • $y = 0$

▶ $x = \sqrt{a^2-y^2}$, syarat $a^2 - y^2 \geq 0$

$x = 0 \to a^2 = y^2$ $\qquad a^2 \geq y^2$

$\sqrt{a^2} = |y|$ $\qquad |y| \leq a$

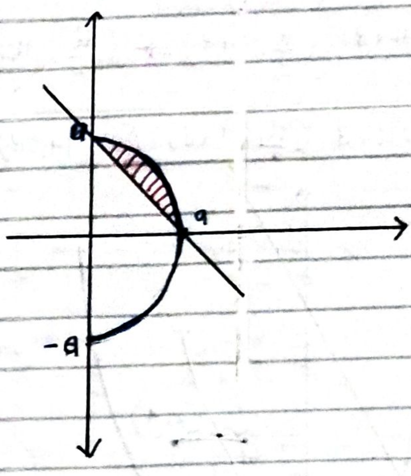
$y = \pm a$ $\qquad -a \leq y \leq a$

$y = 0 \to x = a$

▶ $x = a - y$

$x = 0 \to y = a$

$y = 0 \to x = a$



b). $\displaystyle\int_0^a \left[\int_{a-y}^{\sqrt{a^2-y^2}} dx\right] dy = \int_0^a \left[x\right]_{a-y}^{\sqrt{a^2-y^2}} dy$

$\displaystyle = \int_0^a \sqrt{a^2-y^2} - a + y \; dy$

$\displaystyle = \int_0^a y\,dy - \int_0^a a\,dy + \int_0^a \sqrt{a^2-y^2}\,dy$

* $\displaystyle\int_0^a y\,dy = \frac{1}{2}y^2\Big]_0^a = \frac{a^2}{2}$

* $\displaystyle\int_0^a a\,dy = ay\Big]_0^a = a^2$

* $\displaystyle\int_0^a \sqrt{a^2-y^2}\,dy$

misal : $y = a\sin\theta$

$dy = a\cos\theta\,d\theta$

BA : $y = a\sin\theta$, $\sin\theta = 1$, $\theta = \pi/2$

BB : $\sin\theta = 0$, $\theta = 0$

$\displaystyle = \int_0^{\pi/2} \sqrt{a^2 - (a\sin\theta)^2}\cdot(a\cos\theta)\,d\theta$

$\displaystyle = \int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$\displaystyle = \int_0^{\pi/2} \sqrt{a^2(1-\sin^2\theta)}\; a\cos\theta\,d\theta$

$\displaystyle = \int_0^{\pi/2} \sqrt{a^2\cos^2\theta}\; a\cos\theta\,d\theta$

$\displaystyle = a^2\int_0^{\pi/2} \cos^2\theta\,d\theta$

* $\cos 2\theta = 2\cos^2\theta - 1$ BI

$\displaystyle \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\displaystyle = a^2\left[\int_0^{\pi/2} \frac{1}{2}\,d\theta + \int_0^{\pi/2} \frac{\cos 2\theta}{2}\,d\theta\right]$

$\displaystyle = a^2\left[\left[\frac{1}{2}\theta\right]_0^{\pi/2} + \frac{1}{4}\sin 2\theta\Big]_0^{\pi/2}\right]$

$\displaystyle \rightsquigarrow \int_0^a \sqrt{a^2-y^2}\,dy = a^2\left[\frac{\pi}{4} + \frac{1}{4}(0-0)\right]$

$\displaystyle = \frac{\pi a^2}{4}$

$\displaystyle \Rightarrow \int_0^a\left[\int_{a-y}^{\sqrt{a^2-y^2}} dx\right]dy = \frac{a^2}{2} - a^2 + \frac{\pi a^2}{4}$

$\displaystyle = \frac{\pi a^2 - 2a^2}{4} = \frac{a^2(\pi-2)}{4}$ //

Jawab = $\dfrac{a^2(\pi-2)}{4}$ //

c). • $x = \sqrt{a^2-y^2}$ • $x = a - y$

$x^2 = a^2 - y^2$ $\quad y = a - x$

$y^2 = a^2 - x^2$

$= \sqrt{a^2-x^2}$

$$\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx = \int_0^a \left[\int_{a-x}^{\sqrt{a^2-x^2}} dy \right] dx$$

$$= \int_0^a \left[\; \right]_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \sqrt{a^2-x^2} - (a-x)\, dx$$

$$= \int_0^a \sqrt{a^2-x^2}\, dx - \int_0^a a\, dx + \int_0^a x\, dx$$

* $\int_0^a \sqrt{a^2-x^2} dx =$

misal : $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

BB : $0 = \sin\theta$

$\theta = 0$

BA : $a = a\sin\theta$

$\sin\theta = 1$

$\theta = \pi/2$

$$\int_0^{\pi/2} \sqrt{a^2-(a\sin\theta)^2}\, .\, a.\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\, .\, a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{a^2(1-\sin^2\theta)}\, .\, a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} a\cos\theta\, .\, a.\cos\theta\, d\theta$$

$$= a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

* $\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= a^2 \int_0^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2} \right) d\theta$$

$$= a^2 \left[\int_0^{\pi/2} \frac{1}{2} d\theta + \int_0^{\pi/2} \frac{\cos 2\theta}{2} d\theta \right]$$

$$= a^2 \left[\left[\frac{1}{2}\theta\right]_0^{\pi/2} + \left.\frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \right]$$

$$= a^2 \left[\frac{\pi}{4} + [0-0] \right]$$

$$= \frac{\pi a^2}{4}$$

* $\int_0^a a\, dx = ax\Big]_0^a$

$= a^2$

* $\int_0^a x\, dx = \frac{1}{2}x^2\Big]_0^a$

$= \frac{a^2}{2}$

$$\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx = \int_0^a \sqrt{a^2-x^2}\, dx - \int_0^a a\, dx + \int_0^a x\, dx$$

$$= \frac{\pi a^2}{4} - a^2 + \frac{a^2}{2}$$

$$= \frac{\pi a^2}{4} - \frac{a^2}{2}$$

$$= \frac{\pi a^2 - 2a^2}{4}$$

$$= \frac{a^2(\pi - 2)}{4} //$$

10. $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$

a. Sketsa daerah integrasinya.

- $x = y$
- $x = \sqrt{4-y^2} \rightarrow x^2 = 4-y^2$
  $\downarrow$
  $y^2+x^2 = 4$

$x = 0 \rightarrow 4-y^2 = 0$

$y^2 = 4$

$y = 2 \vee y = -2$

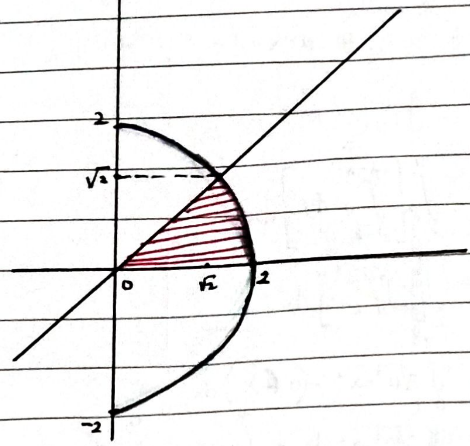
$y = 0 \rightarrow x = 2$

- $x = y \rightarrow y = \sqrt{4-y^2}$

$y^2 = 4-y^2$

$y^2 = 2$

$y = \sqrt{2}$



b. Dapatkan nilai dari integral tersebut

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy = \int_0^{\sqrt{2}}\left[\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\right]dy$$

$$= \int_0^{\sqrt{2}}\left[\int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{(y^2+1)}^2+x^2}\,dx\right]dy$$

※ Ingat! $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$, maka :

$$= \int_0^{\sqrt{2}}\left[\frac{1}{\sqrt{y^2+1}}\cdot\tan^{-1}\frac{x}{\sqrt{y^2+1}}\Bigg]_y^{\sqrt{4-y^2}}\right]dy$$

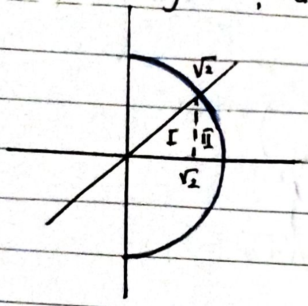
$$= \int_0^{\sqrt{2}}\left[\frac{1}{\sqrt{y^2+1}}\left(\tan^{-1}\frac{\sqrt{4-y^2}}{\sqrt{y^2+1}} - \tan^{-1}\frac{y}{\sqrt{y^2+1}}\right)\right]dy$$

$$= \int_0^{\sqrt{2}} \frac{\tan^{-1}\left(\frac{\sqrt{4-y^2}}{\sqrt{y^2+1}}\right) - \tan^{-1}\frac{y}{\sqrt{y^2+1}}}{\sqrt{y^2+1}}\,dy$$

(Karena susah, kita coba tukar urutan integralnya).

▶ Tukar Urutan Integral

* Bagi daerah menjadi 2, atau jika digambar :



▶ Integral → $L_{tot} = L_1 + L_2$

Luas daerah I

→ Berjalan pada daerah y (vertikal) akan ada 2 persamaan $\begin{cases} \text{atas} : y = x \\ \text{bawah} : y = 0 \end{cases}$

→ Berjalan pada daerah x (horizontal) akan ada dua persamaan $\begin{cases} : x = \sqrt{2} \\ \text{kiri} : x = 0 \end{cases}$

maka integralnya :

$$\int_0^{\sqrt{2}} \int_0^x \frac{1}{1+x^2+y^2} \, dx \, dy = \int_0^{\sqrt{2}} \left[\int_0^x \frac{1}{(\sqrt{1+x^2})^2 + y^2} \, dy \right] dx$$

* Ingat ! $\int \frac{dy}{a^2+y^2} = \frac{1}{a} \tan^{-1} \frac{y}{a} + C$, maka :

$$= \int_0^{\sqrt{2}} \left[\left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^x \right] dx$$

$$= \int_0^{\sqrt{2}} \left[\frac{1}{\sqrt{1+x^2}} \left(\tan^{-1} \frac{x}{\sqrt{1+x^2}} - \tan^{-1}(0) \right) \right] dx$$

$$= \int_0^{\sqrt{2}} \frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1} \frac{x}{\sqrt{1+x^2}} \, dx$$

(Karena susah diintegralkan, maka gunakan cara lain).

↳ Ubah ke bentuk Kutub

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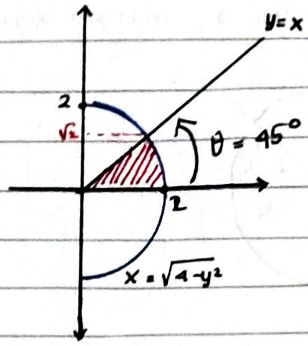
10. $$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$$

a. Sketsa Daerah Integrasinya !



- $x = y$
- $x = \sqrt{4-y^2}$

$x^2 = 4 - y^2$

$x^2 + y^2 = 4$ (maka setengah lingkaran pada sumbu-x positif).

$r = 2$, pusat $(0,0)$.

b. Dapatkan nilai dari integral tersebut !

$$F(x) = \frac{1}{1+x^2+y^2}$$

* Ingat ! $x = r\cos\theta$ $\quad y = r\sin\theta$

$$G(r,\theta) = \frac{1}{1+r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \frac{1}{1+r^2[\cos^2\theta+\sin^2\theta]}$$

$$= \frac{1}{1+r^2}$$

$$\int_0^{\pi/4}\int_0^{2} \frac{1}{1+r^2}\,dr\,.\,d\theta = \int_0^{\pi/4}\left[\int_0^2 \frac{1}{1+r^2}\,dr\right]d\theta$$

$$= \int_0^{\pi/4}\left[\tan^{-1} r\right]_0^2 d\theta$$

$$= \int_0^{\pi/4}\left(\tan^{-1}(2) - \tan^{-1}(0)\right)d\theta$$

$$= \int_0^{\pi/4} \tan^{-1} 2\, d\theta$$

$$= \theta\,.\,\tan^{-1}(2)\Big]_0^{\pi/4}$$

$$= \frac{\pi}{4}\tan^{-1}(2)$$

Jawab $= \dfrac{\pi\,.\,\tan^{-1}(2)}{4}$

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